

Mesoscopic rings with Spin-Orbit interactions

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Abstract. A didactic description of charge and spin equilibrium currents on mesoscopic rings in the presence of Spin-Orbit interaction is presented. Emphasis is made on the non trivial construction of the correct Hamiltonian in polar coordinates, the calculation of eigenvalues and eigenfunctions and the symmetries of the ground state properties. Spin currents are derived following an intuitive definition and then a more thorough derivation is built upon the canonical Lagrangian formulation that emphasizes the SU(2) gauge structure of the transport problem of spin 1/2 fermions in spin-orbit active media. The quantization conditions that follow from the constraint of single-valued Pauli spinors are also discussed. The targeted students are those of a graduate Condensed Matter Physics course.

Résumé. On donne une description didactique des courants de charge et de spin dans un anneau mésoscopique en présence de couplage spin-orbite. L'accent est mis sur la construction de l'hamiltonien, le calcul des valeurs propres et des vecteurs propres et les propriétés liées à la symétrie par renversement temporel. Le courant de spin est calculé à partir d'une définition intuitive, puis une démonstration plus approfondie est construite à partir de la formulation lagrangienne qui met en évidence la structure de jauge SU(2). Les conditions de quantification qui découlent des contraintes de raccordement des spineurs de Pauli sont également discutées. Le public visé est celui d'un cours avancé de physique de la matière condensée.

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73.23.-b Electronic transport in mesoscopic systems,

85.75.-d Magnetoelectronics; spintronics: devices exploiting spin polarized transport or integrated magnetic fields,

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1. Introduction

The study of spin transport properties in materials, especially in semi-conductors where two-dimensional electron gases (2DEG) can be fabricated, has become an important field of research both theoretically and experimentally in the last decade. One of the key concept there is the Spin-Orbit (SO) interaction which allows for spin manipulations. Mesoscopic rings are very simple laboratories for the investigation of various quantum effects [1] in the presence of spin-orbit interactions. They are the subject of intensive interest, both for fundamental reasons and potential applications

in spintronics devices. Let us only mention a few fundamental studies in connection with phase effects, such as Aharonov-Casher effect [2,3] or Sagnac phase shifts [4], spin interferences and spin filtering [5–9] or in the fashionable subject of graphene band structure [10]. On the more applied side, we can cite spin manipulations [11–14], studies of spin related transport properties [15–17] and persistent currents [18–20], spin filtering in arrays of SO quantum rings [21] that operate on principles of spin interferometry analogous to their optical counterparts [22] and concepts for the elaboration of qubits gates [23].

Many of the concepts involved in spin transport are rooted in very basic quantum mechanics but in contexts that may be unfamiliar for the beginner, such as multiply connected geometries like closed loops or rings. This paper will serve to sort out such uncommon applications opening the scope of understanding forefront topics such as spintronics.

In this paper, we present an analysis of spin transport in a mesoscopic ring. The exposition is suitable for a course in condensed matter physics at the graduate level. We carefully present the details of the non trivial construction of the correct Hamiltonian in polar coordinates, the calculation of eigenvalues and eigenfunctions and the symmetries of the ground state properties with respect to time reversal symmetry. Charge and spin currents are derived following an intuitive and heuristic definition and then a more thorough derivation is built upon the canonical Lagrangian formulation that emphasizes the $SU(2)$ gauge structure of the transport problem. The path integral approach and the quantization conditions that follow from the constraint of single-valued Pauli spinors are also discussed for the benefit of a full picture that will be valuable to the student.

The Spin-Orbit interaction finds its origin in the Pauli equation, which follows from the non-relativistic limit of the Dirac equation,

$$\begin{aligned} \mathbf{H} = & \left[\frac{(\vec{p} - e\vec{A})^2}{2m} - e\phi \right] \mathbf{1}_{2 \times 2} - \left[\frac{\vec{p}^4}{8m^3c^2} - \frac{e\hbar^2}{8m^2c^2} \vec{\nabla} \cdot \vec{E} \right] \mathbf{1}_{2 \times 2} \\ & - \frac{e\hbar}{2m} \vec{\sigma} \cdot \vec{B} + \frac{e\hbar \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \times \vec{E}}{4m^2c^2}. \end{aligned} \quad (1)$$

Here we consider electrons, $e = -|e|$. Ordinary vectors are denoted by arrows and bold face characters are used for 2×2 matrices. The first term in the first line corresponds to the usual Schrödinger equation including the kinetic energy with a minimal coupling to the electromagnetic gauge field \vec{A} and a scalar potential contribution $-e\phi$. The second term in the first line describes the first relativistic correction to the kinetic energy and the Darwin term, where \vec{E} is the electric field and c the speed of light. These first two terms are proportional to the 2×2 identity matrix in spin space $\mathbf{1}_{2 \times 2}$. The second line comprises explicitly spin-dependent terms, first the Zeeman interaction [24] where \vec{B} is the magnetic field and $\vec{\sigma}$ is the vector of Pauli matrices [24] and the second term is the Spin-Orbit interaction, written with the minimal coupling to the gauge vector. We have assumed a static potential so that the rotor of the electric field is absent and the Spin-Orbit interaction is limited to the term mentioned here [25].

Forgetting about the magnetic field, the Spin-Orbit interaction has a simple interpretation in terms of the interaction of the spin magnetic moment of the particles, (here supposed to be electrons of Landé factor [26] $g \simeq 2$ and spin $\vec{s} = \frac{1}{2}\hbar\vec{\sigma}$), $\vec{\mu} = g\frac{e}{2m}\vec{s} = -\frac{|e|\hbar}{2m}\vec{\sigma}$, with the magnetic field produced by all external moving charges in the electron rest frame. Assuming a uniform electric field \vec{E} acting on the moving electrons, the SO contribution to the Hamiltonian results from the interaction

$-\vec{\mu} \cdot \vec{B}_{\text{rest fr.}}$ of $\vec{\mu}$ with the effective magnetic field experienced by the particles in their rest frame. In the case of a Lorentz change of reference frame, one has

$$\vec{B}_{\text{rest fr.}} = \frac{1}{c^2}(-\vec{v}) \times \vec{E} = -(mc^2)^{-1}(\vec{p} \times \vec{E}), \quad (2)$$

where \vec{v} and \vec{p} refer to the dynamical variables of the electron. This expression is corrected by a similar contribution with a factor $-\frac{1}{2}$ due to the Thomas precession in the case of a closed orbit. Such a term appears when the proper Lorentz transformation for the fields is considered [26]. In such a way, the resulting SO interaction is usually written (e.g. in atoms)

$$\begin{aligned} H_{\text{SO}} &= -\frac{|e|\hbar}{2m^2c^2} \vec{\sigma} \cdot (\vec{p} \times \vec{E}), \\ &= \frac{|e|}{m^2c^2} \frac{1}{r} \frac{\partial \phi(r)}{\partial r} \vec{s} \cdot \vec{L}, \end{aligned} \quad (3)$$

where we have substituted a spherically symmetric potential, $\vec{E} = -\vec{\nabla}\phi = \frac{1}{r} \frac{\partial \phi(r)}{\partial r} \vec{r}$ and the interaction is proportional to $\vec{s} \cdot \vec{L}$. It is clear now why, in the context of atoms, the interaction is christened the Spin-Orbit interaction i.e. \vec{L} pertains to the angular momentum of the orbit while \vec{s} pertains to intrinsic angular momentum, the spin.

The prefactor in this expression depends on the details of the problem (Landé factor, effective mass, ...) its sign even depends on the nature of the particles involved. In semiconductor physics, one usually introduces the notations α or β for this coefficient, which is determined perturbatively in the $\vec{k} \cdot \vec{p}$ theory by a matrix element in the Bloch wave functions basis. In the case of a two-dimensional electron gas with a gate voltage applied perpendicular to the 2d sample (with a non-symmetric confining potential generating a space inversion asymmetry (SIA)), this term is known as the Rashba SO interaction [27, 28],

$$\mathbf{V}_{\text{Rashba}} = \text{const.} \cdot \vec{\sigma} \cdot (\vec{E} \times \vec{p}) = \alpha(\sigma_y p_x - \sigma_x p_y). \quad (4)$$

Note that the Rashba SO amplitude can be tuned experimentally using a gate voltage, since the prefactor α is proportional to the electric field (see figure 1).

Gate Voltage

\vec{E}

2D Electron gas

Figure 1. The typical Rashba situation with an electric field, created by a gate voltage, perpendicular to a ring fabricated on a two dimensional electron gas.

The calculation of matrix elements leading to α is generally a hard task and one usually uses phenomenological expressions compatible with the crystal symmetries or

computes them directly from experiment [29]. The Rashba interaction applies in the case of SIA. When there is bulk inversion asymmetry (BIA) i.e. the crystal unit cell lacks inversion symmetry, we have the Dresselhaus ‘flavor’ of the SO interaction in $3d$ systems [30],

$$\mathbf{V}_{D,3d} = \text{const. } k_x(k_y^2 - k_z^2)\boldsymbol{\sigma}_x + \text{c.p.} \quad (5)$$

where c.p. stands for cyclic permutations. In the case of electrons confined in two dimensions, the expectation value along the confinement, quantized dimension should be considered. If that direction is z , then $\langle k_z \rangle \simeq 0$, $\langle k_z^2 \rangle \simeq (\pi/\ell)^2$, ℓ being the typical confinement length or spatial width of the confinement potential. The Dresselhaus SO interaction thus takes the simple form, here written in a notation closer to equation (4),

$$\mathbf{V}_{\text{Dresselhaus}} = \beta(\boldsymbol{\sigma}_x p_x - \boldsymbol{\sigma}_y p_y), \quad (6)$$

where we neglect cubic terms in k and the average values computed are lumped in the definition of β . For more details on SO interactions in semi-conductors, see Refs. [30–32].

2. Derivation of the ring Hamiltonian

2.1. The argument of Meijer, Morpurgo and Klapwijk for the Rashba SO interaction

To make the discussion simpler, we neglect any magnetic field effect, which is well discussed in the literature. Here we discuss the non-trivial point of designing a Hermitian Hamiltonian when it is not in cartesian coordinates. We explain in detail such construction so the student can understand the pitfalls generally avoided in quantum mechanics courses because of a preferential system of coordinates. Many researchers overlooked this hermiticity problem until the recent paper by Meijer, Morpurgo and Klapwijk [33].

The classical argument in reference [33] is essentially the following. We consider a 2DEG, neglecting interactions between electrons and add the Rashba SO interaction in cylindrical coordinates (ρ, φ, z) ,

$$\mathbf{H}_{2d} = -\frac{\hbar^2}{2m} (\partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\partial_\varphi^2) \mathbf{1}_{2 \times 2} \quad (7)$$

$$+ \alpha \rho^{-1} (\boldsymbol{\sigma}_x \cos \varphi + \boldsymbol{\sigma}_y \sin \varphi) i \hbar \partial_\varphi + \alpha (\boldsymbol{\sigma}_x \sin \varphi - \boldsymbol{\sigma}_y \cos \varphi) i \hbar \partial_\rho.$$

The Rashba SO interaction in the second line comes from equation (4) with the definitions $p_x = -i\hbar\partial_x$ and $p_y = -i\hbar\partial_y$ after the substitutions $\partial_x = -(\rho)^{-1}\sin\varphi\partial_\varphi + \cos\varphi\partial_\rho$ and $\partial_y = (\rho)^{-1}\cos\varphi\partial_\varphi + \sin\varphi\partial_\rho$. Fixing the radial distance ρ to the ring radius a and neglecting the radial derivatives leads to a “1d” Hamiltonian (called \mathbf{H}'_{1d} for the moment) for the ring,

$$\mathbf{H}'_{1d} = \frac{\hbar^2}{2ma^2} (i\partial_\varphi)^2 + \alpha \hbar a^{-1} (\boldsymbol{\sigma}_x \cos \varphi + \boldsymbol{\sigma}_y \sin \varphi) i \partial_\varphi. \quad (8)$$

In [33], the authors note that the last term in (8) is non-Hermitian, since

$$\begin{aligned} \langle F | (\boldsymbol{\sigma}_x \cos \varphi + \boldsymbol{\sigma}_y \sin \varphi) i \partial_\varphi | G \rangle^* &= \left(\int_0^{2\pi} d\varphi (F_\uparrow^* \ F_\downarrow^*) \begin{pmatrix} 0 & ie^{-i\varphi}\partial_\varphi \\ ie^{i\varphi}\partial_\varphi & 0 \end{pmatrix} \begin{pmatrix} G_\uparrow \\ G_\downarrow \end{pmatrix} \right)^* \\ &= [-iF_\uparrow e^{i\varphi} G_\downarrow^* - iF_\downarrow e^{-i\varphi} G_\uparrow^*]_0^{2\pi} + \int_0^{2\pi} d\varphi (G_\uparrow^* \ G_\downarrow^*) \begin{pmatrix} 0 & ie^{-i\varphi}\partial_\varphi \\ ie^{i\varphi}\partial_\varphi & 0 \end{pmatrix} \begin{pmatrix} F_\uparrow \\ F_\downarrow \end{pmatrix} \\ &\quad + \int_0^{2\pi} d\varphi (G_\uparrow^* \ G_\downarrow^*) \begin{pmatrix} 0 & e^{-i\varphi} \\ -e^{i\varphi} & 0 \end{pmatrix} \begin{pmatrix} F_\uparrow \\ F_\downarrow \end{pmatrix}. \\ &= \langle G | (\boldsymbol{\sigma}_x \cos \varphi + \boldsymbol{\sigma}_y \sin \varphi) i \partial_\varphi | F \rangle - i \langle G | (\boldsymbol{\sigma}_x \sin \varphi - \boldsymbol{\sigma}_y \cos \varphi) | F \rangle, \end{aligned} \quad (9)$$

where $|F\rangle$ and $|G\rangle$ are Pauli spinors, and the integrated terms vanish if we impose single-valued spinors, i.e. $\langle\varphi + 2\pi|F\rangle = \langle\varphi|F\rangle$. From the property given in equation (9), we may infer that a Hermitian operator could be formed by the combination $\mathbf{V} = \alpha\hbar a^{-1}\mathbf{W}$ where

$$\mathbf{W} = (\boldsymbol{\sigma}_x \cos \varphi + \boldsymbol{\sigma}_y \sin \varphi)i\partial_\varphi - iA(\boldsymbol{\sigma}_x \sin \varphi - \boldsymbol{\sigma}_y \cos \varphi), \quad (10)$$

since

$$\begin{aligned} \langle F|\mathbf{W}|G\rangle^* &= \langle G|(\boldsymbol{\sigma}_x \cos \varphi + \boldsymbol{\sigma}_y \sin \varphi)i\partial_\varphi|F\rangle \\ &\quad - i(1-A^*)\langle G|(\boldsymbol{\sigma}_x \sin \varphi - \boldsymbol{\sigma}_y \cos \varphi)|F\rangle, \end{aligned} \quad (11)$$

and

$$\begin{aligned} \langle G|\mathbf{W}|F\rangle &= \langle G|(\boldsymbol{\sigma}_x \cos \varphi + \boldsymbol{\sigma}_y \sin \varphi)i\partial_\varphi|F\rangle \\ &\quad - iA\langle G|(\boldsymbol{\sigma}_x \sin \varphi - \boldsymbol{\sigma}_y \cos \varphi)|F\rangle. \end{aligned} \quad (12)$$

The r.h.s. of equations (11) and (12) are equal provided that $A^* = A = \frac{1}{2}$. This value in equation (10) gives the correct expression that renders the Rashba SO interaction on the ring Hermitian.

The method used in Ref. [33] to build hermiticity into the SO Hamiltonian, consists in the introduction of a confining potential $V(\rho)$ in equation (7) in order to localize the particle on a circle of radius a . If this potential is steep enough, the particles will lie in the ground state $R_0(\rho)$ of

$$H_\rho = \text{KE} + V(\rho) = -\frac{\hbar^2}{2m}(\partial_\rho^2 + \rho^{-1}\partial_\rho) + V(\rho), \quad (13)$$

where KE is the contribution of the radial motion to the kinetic energy and the correct 1d Hamiltonian ($\mathbf{H}_{\text{Rashba}}^{\text{ring}}$) is defined by the projection, on this ground state, of the angular dependence of \mathbf{H}_{2d} ,

$$\mathbf{H}_{\text{Rashba}}^{\text{ring}} = \langle R_0|\mathbf{H}_{2d} - \text{KE} \mathbf{1}_{2 \times 2}|R_0\rangle, \quad (14)$$

where $|R_0\rangle$ is extended to a two-component object. Note the very peculiar situation in polar coordinates, KE does not simply coincide with $p_\rho^2/2m$ (see e.g. Ref. [34] for a discussion on this unrecognized point). The calculation is now made specific by considering a circular Harmonic potential $V(\rho) = \frac{1}{2}m\omega^2(\rho - a)^2$ and by the explicit calculation of the matrix elements $\langle R_0|\rho^{-1}|R_0\rangle = a^{-1}$ and $\langle R_0|\partial_\rho|R_0\rangle = -(2a)^{-1}$. The resulting Hamiltonian $\mathbf{H}_{\text{Rashba}}^{\text{ring}}$ then follows from the substitutions $\rho \rightarrow a$, $\partial_\rho \rightarrow -(2a)^{-1}$ in $\mathbf{H}_{2d} + \frac{\hbar^2}{2m}(\partial_\rho^2 + \rho^{-1}\partial_\rho)$. Thus

$$\mathbf{H}_{\text{Rashba}}^{\text{ring}} = \frac{\hbar^2}{2ma^2}(i\partial_\varphi)^2 \mathbf{1}_{2 \times 2} \quad (15)$$

$$+ \alpha\hbar a^{-1}(\boldsymbol{\sigma}_x \cos \varphi + \boldsymbol{\sigma}_y \sin \varphi)i\partial_\varphi - i\alpha\hbar(2a)^{-1}(\boldsymbol{\sigma}_x \sin \varphi - \boldsymbol{\sigma}_y \cos \varphi).$$

It can be checked that the Hamiltonian is now Hermitian, since it coincides with the choice $A^* = A = \frac{1}{2}$, according to the procedure outlined in Eqs. (10-12).

2.2. Symmetrization of the original Hamiltonian

Although the previous derivation is of course correct, it is instructive to propose another, probably more straightforward derivation. The SO interaction in cylindrical coordinates with a perpendicular electric field, $\vec{\alpha} = \alpha\vec{e}_z$, takes the form

$$\vec{\sigma} \cdot (\vec{\alpha} \times \vec{p}) = -\alpha(\boldsymbol{\sigma}_\rho p_\varphi - \boldsymbol{\sigma}_\varphi p_\rho), \quad (16)$$

with $\boldsymbol{\sigma}_\rho = \boldsymbol{\sigma}_x \cos \varphi + \boldsymbol{\sigma}_y \sin \varphi$ and $\boldsymbol{\sigma}_\varphi = -\boldsymbol{\sigma}_x \sin \varphi + \boldsymbol{\sigma}_y \cos \varphi$ treated for the moment as classical variables (i.e. we forget about possible commutation problems). In order

to make the corresponding expression Hermitian in Quantum Mechanics, one has to use the correspondence principle, i.e. add inequivalent orders of non-commuting operators, and symmetrize the “classical” expression, using also $p_\rho = -i\hbar(\partial_\rho - (2\rho)^{-1})$ and $p_\varphi = -i\hbar a^{-1}\partial_\varphi$. Let us also note that when the electrons are confined on a ring, then $\langle p_\rho \rangle = \langle R_0 | p_\rho | R_0 \rangle = 0$ (and not $\langle R_0 | \partial_\rho | R_0 \rangle = 0$ as it was done to get equation (8)) and $\rho = a$. This leads to

$$\mathbf{V}_{Rashba}^{\text{ring}} = -\frac{1}{2}\alpha\{\boldsymbol{\sigma}_\rho, p_\varphi\} = i\hbar\alpha a^{-1}(\boldsymbol{\sigma}_\rho\partial_\varphi + \frac{1}{2}\partial_\varphi\boldsymbol{\sigma}_\rho), \quad (17)$$

where $\{\boldsymbol{\sigma}_\rho, p_\varphi\}$ denotes the anticommutator $\boldsymbol{\sigma}_\rho p_\varphi + p_\varphi \boldsymbol{\sigma}_\rho$. Using $\partial_\varphi \boldsymbol{\sigma}_\rho = \boldsymbol{\sigma}_\varphi$, equation (15) eventually follows,

$$\mathbf{H}_{Rashba}^{\text{ring}} = \frac{\hbar^2}{2ma^2}(i\partial_\varphi)^2 \mathbf{1}_{2 \times 2} + i\hbar\alpha a^{-1}(\boldsymbol{\sigma}_\rho\partial_\varphi + \frac{1}{2}\boldsymbol{\sigma}_\varphi). \quad (18)$$

Let us emphasize on the fact that the condition $\langle p_\rho \rangle = 0$ is equivalent to Meijer et al’s derivation, since it implies $-i\hbar\langle R_0 | \partial_\rho - (2\rho)^{-1} | R_0 \rangle = 0$ with a radial wave function strongly localized on $\rho \simeq a$. Our proposed derivation, nevertheless, has the merit of being simpler and more general, since it does not rely on any particular assumption on the confining potential (except the fact that it enforces the particles to move along a ring).

2.3. Hamiltonian in the rotated spin basis

In the previous sections, the Rashba Hamiltonian (18), initially expressed in Cartesian coordinates, has been rewritten in polar coordinates. To preserve the hermiticity of the Hamiltonian, it has been shown that an extra term should be added if the expression is not properly symmetrized prior to quantization. In the following, we give a more physical picture that gives rise to this extra term.

We first start by noting that the change of coordinates $(x, y) \rightarrow (\rho, \varphi)$ induces a rotation by an angle φ of the local frame: $(\vec{e}_x, \vec{e}_y) \rightarrow (\vec{e}_\rho, \vec{e}_\varphi)$. All vector fields, including $\boldsymbol{\sigma}$ are now expressed in this local frame. However, *no such rotation* has been applied to the spinor basis which thus remains the same at any point of the circle. The notations $\boldsymbol{\sigma}_\rho$ and $\boldsymbol{\sigma}_\varphi$ may thus be misleading. Let us introduce such a rotation of the spinor basis:

$$|F'\rangle = e^{i\frac{\varphi}{2}\sigma_z}|F\rangle = (\cos\frac{\varphi}{2}\mathbf{1}_{2 \times 2} + i\sin\frac{\varphi}{2}\sigma_z)|F\rangle. \quad (19)$$

Since the rotation axis is (Oz) , the quantification axis is not changed during the transformation. The two spin components simply get a phase

$$|\uparrow'\rangle = e^{i\frac{\varphi}{2}}|\uparrow\rangle, \quad |\downarrow'\rangle = e^{-i\frac{\varphi}{2}}|\downarrow\rangle. \quad (20)$$

In the new local spinor basis, the Hamiltonian becomes

$$\mathbf{H}' = e^{i\frac{\varphi}{2}\sigma_z}\mathbf{H}e^{-i\frac{\varphi}{2}\sigma_z}. \quad (21)$$

Using the identity $\boldsymbol{\sigma}_\rho\sigma_z = -i\boldsymbol{\sigma}_\varphi$, the correct Hermitian Rashba Hamiltonian can be cast as

$$\mathbf{H}_{Rashba}^{\text{ring}} = \frac{\hbar^2}{2ma^2}(i\partial_\varphi)^2 \mathbf{1}_{2 \times 2} + i\hbar\alpha a^{-1}\boldsymbol{\sigma}_\rho(\partial_\varphi + \frac{i}{2}\sigma_z). \quad (22)$$

Under the change of spinor basis, the new rotated Hamiltonian becomes

$$e^{i\frac{\varphi}{2}\sigma_z}\mathbf{H}_{Rashba}^{\text{ring}}e^{-i\frac{\varphi}{2}\sigma_z} = \frac{\hbar^2}{2ma^2}(i\partial_\varphi\mathbf{1}_{2 \times 2} + \frac{1}{2}\sigma_z)^2 + i\hbar\alpha a^{-1}\sigma_x\partial_\varphi, \quad (23)$$

since $\boldsymbol{\sigma}'_\rho \equiv e^{i\frac{\varphi}{2}\sigma_z}\boldsymbol{\sigma}_\rho e^{-i\frac{\varphi}{2}\sigma_z} = \boldsymbol{\sigma}_x$. Interestingly, after rotation of the spinor basis the potential energy recovers a form $\propto \boldsymbol{\sigma}'_\rho p_\varphi$ similar to the original Rashba Hamiltonian

before symmetrization in (16). From the physical point of view, one indeed expects the potential energy to be the same in all local orthonormal frames. This implies that, starting from the original Rashba Hamiltonian, the change of coordinates should be accompanied by a change of the local vector and spinor frames. The spin-orbit term then keeps the same form. When expressed in the original spin basis, the additional term needed for symmetrization appears naturally. In contradistinction to the potential energy, additional terms may appear in the kinetic energy when one changes the spinor basis. They are interpreted as inertial forces due to the fact that the local frame is not inertial. This is the origin of the extra term $\frac{1}{2}\sigma_z$ in the Hamiltonian (23) (note the appearance of the z component of the total angular momentum, $\mathbf{L}_z + \mathbf{s}_z$). When the particle moves along the circle, the local spinor basis turns by the same angle. Consider for example a constant spinor. To accommodate the change of local basis along the circle, it should have φ -dependent spin-components. Since no kinetic energy is associated to the spin, a term involving derivatives ∂_φ would produce a term which is precisely cancelled by the additional contribution $\frac{1}{2}\sigma_z$.

3. Eigenenergies and Eigenvectors on a ring

3.1. The Rashba SO interaction

Once the correct Hermitian Hamiltonian has been written in polar coordinates we can confidently derive the eigenvalues and eigenvectors for the strictly one dimensional ring. This section will benefit the student by explicitly obtaining the eigenfunctions and eigenvalues of the system, from which we can obtain all observables or actually measurable quantities in experiments. We will also make some basic symmetry considerations that will illuminate details of why the wavefunction and energies take their particular form.

First we rewrite the Hamiltonian in a clever way in order to arrive rapidly at the eigenvalues and introduce physical insight by pointing out the existence of a gauge field associated to topological phenomena connected to spin transport. An alternative form of the Hamiltonian (18) was given in Ref. [35]. We observe that

$$\left(i\partial_\varphi \mathbf{1}_{2 \times 2} + \frac{ma\alpha}{\hbar} \boldsymbol{\sigma}_\rho\right)^2 = (i\partial_\varphi)^2 \mathbf{1}_{2 \times 2} + \frac{2ma\alpha}{\hbar} \boldsymbol{\sigma}_\rho (i\partial_\varphi) + \frac{ma\alpha}{\hbar} i\boldsymbol{\sigma}_\varphi + \left(\frac{ma\alpha}{\hbar}\right)^2 \mathbf{1}_{2 \times 2}. \quad (24)$$

The factor of two in the second term on the right accounts for the appropriate action of an derivative operator. Such form allows one to recast the Hamiltonian as

$$\mathbf{H}_{Rashba}^{\text{ring}} = \frac{\hbar^2}{2ma^2} \left[\left(i\partial_\varphi \mathbf{1}_{2 \times 2} + \frac{ma\alpha}{\hbar} \boldsymbol{\sigma}_\rho\right)^2 - \left(\frac{ma\alpha}{\hbar}\right)^2 \mathbf{1}_{2 \times 2} \right]. \quad (25)$$

In order to find the eigenstates of the Hamiltonian (25), one first solves the eigenvalue equation

$$\left(i\partial_\varphi \mathbf{1}_{2 \times 2} + \frac{ma\alpha}{\hbar} \boldsymbol{\sigma}_\rho\right) \Psi = \varepsilon \Psi, \quad (26)$$

or

$$\begin{pmatrix} i\partial_\varphi & \frac{ma\alpha}{\hbar} e^{-i\varphi} \\ \frac{ma\alpha}{\hbar} e^{i\varphi} & i\partial_\varphi \end{pmatrix} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}. \quad (27)$$

It follows that the general form for the eigenfunctions should be

$$\Psi_{n,s}^\lambda = e^{i\lambda n\varphi} \begin{pmatrix} A_{\lambda,s} e^{-i\varphi/2} \\ B_{\lambda,s} e^{i\varphi/2} \end{pmatrix}, \quad (28)$$

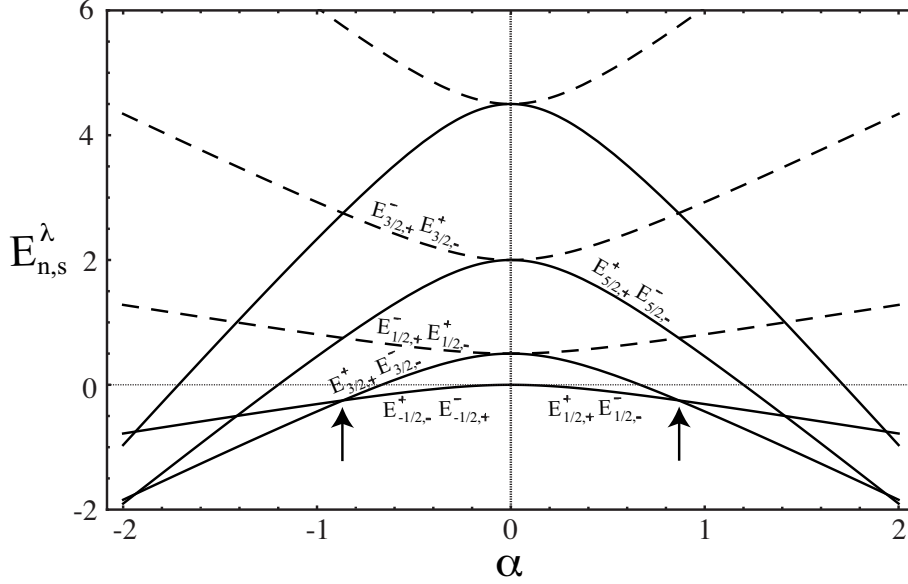


Figure 2. Energies of electrons on a ring with spin-orbit interaction as a function of the SO interaction strength α . The energy is in units of \hbar^2/ma^2 while α is in units of \hbar/ma . Indicated are the eigenvalue labels for the first five levels (note the ground state is fourfold degenerate). The graph shows how the limit $\alpha = 0$ is fourfold degenerate as expected since the time inversion symmetry turns into independent space and spin inversion symmetry. Arrows indicate change in lowest energy level.

where n is the main quantum number and we account for the two eigenvalues that will arise, $s = \pm$, for each wave propagation direction $\lambda = \pm$. So, assuming such a wavefunction we obtain the corresponding eigenvalues

$$\varepsilon_{n,s}^{\lambda} = \frac{s}{2} \sqrt{1 + 4 \left(\frac{ma\alpha}{\hbar} \right)^2} - \lambda n, \quad (29)$$

where n is a half odd integer as will be seen below when we address the wavefunctions. The eigenvalues of (25) then simply follow

$$E_{n,s}^{\lambda} = \frac{\hbar^2}{2ma^2} \left[\left(n - \frac{\lambda s}{2} \sqrt{1 + 4 \left(\frac{ma\alpha}{\hbar} \right)^2} \right)^2 - \left(\frac{ma\alpha}{\hbar} \right)^2 \right]. \quad (30)$$

Note that left and right propagating waves with the same s -index are not degenerate but time reversal symmetry is satisfied i.e. $E_{n,s}^{\lambda} = E_{n,-s}^{-\lambda}$ (simultaneous change of λ and s). This symmetry reflects the fact that the SO interaction is not space inversion symmetric, but only time reversal symmetric. When the spin-orbit interaction is absent ($\alpha = 0$), space and spin inversion symmetry is recovered and energies of both clockwise and counterclockwise modes are degenerate independent of the spin label. Figure 2 shows the energy levels as a function of the SO strength. The ordering of the levels are indicated according to the spin orientation and sense of the current. The free electron on a ring is recovered for all levels as one can see from the fourfold degeneracy at $\alpha = 0$, rendering all possible combinations of the values of λ, s for a fixed integer value of N in the relation $E_{n,s}^{\lambda} = \hbar^2(n - \lambda s/2)^2/2ma^2 = \hbar^2 N^2/2ma^2$

Table 1. Energies (in units of $\hbar^2/2ma^2$) for the first three levels and their degeneracies according to values of the quantum numbers n , λ and s in the limit of zero spin orbit, and the corresponding values of the free electron integer quantum number N . Also shown is how degeneracies occur when the SO interaction is turned on leaving only time reversal symmetry.

E ($\hbar^2/2ma^2$)	n	λ	s	N	$\alpha \neq 0$
0	-1/2	+	-	0	deg
0	-1/2	-	+	0	
0	1/2	+	+	0	
0	1/2	-	-	0	
1	1/2	+	-	1	deg
1	1/2	-	+	1	
1	3/2	+	+	1	deg
1	3/2	-	-	1	
4	3/2	+	-	2	deg
4	3/2	-	+	2	
4	5/2	+	+	2	deg
4	5/2	-	-	2	

(with n half odd integers). Table 1 shows the energies for the first three levels in the limit of $\alpha = 0$ and how degeneracies are broken when the SO interaction is turned on.

Now we compute the eigenfunctions. In order to deal with the subtleties associated with the change in sign of spin and propagation direction we explicitly do the $\lambda = +$ case. Using equations (27) and (28) we find from the secular equation

$$B_{+,s} = \frac{\hbar}{2ma\alpha} \left(\frac{s}{\cos\theta} - 1 \right) A_{+,s}, \quad (31)$$

where $\cos\theta = 1/\sqrt{1+4(ma\alpha/\hbar)^2}$. In order to conform to a canonical spinor we choose $A_{+,+} = \cos\theta/2$, and thus $B_{+,+} = \sin\theta/2$ for normalization, so we have

$$\frac{\hbar}{2ma\alpha} \left(\frac{1 - \cos\theta}{\cos\theta} \right) \cos\frac{\theta}{2} = \sin\frac{\theta}{2}, \quad (32)$$

from which we have the condition

$$\tan\theta = \frac{2ma\alpha}{\hbar}. \quad (33)$$

The choice for the second eigenfunction is $A_{+,-} = -\sin\theta/2$, and doing the same exercise we arrive at the two eigenspinors

$$\Psi_{n,+}^+ = e^{in\varphi} \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\varphi/2} \\ \sin\frac{\theta}{2}e^{i\varphi/2} \end{pmatrix}, \quad \Psi_{n,-}^+ = e^{in\varphi} \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\varphi/2} \\ \cos\frac{\theta}{2}e^{i\varphi/2} \end{pmatrix} \quad (34)$$

The corresponding eigenfunctions for $\lambda = -$ are

$$\Psi_{n,+}^- = e^{-in\varphi} \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\varphi/2} \\ \sin\frac{\theta}{2}e^{i\varphi/2} \end{pmatrix}, \quad \Psi_{n,-}^- = e^{-in\varphi} \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\varphi/2} \\ \cos\frac{\theta}{2}e^{i\varphi/2} \end{pmatrix} \quad (35)$$

Using the time reversal operator for spin 1/2 particles $\Theta = -i\sigma_y K$ where σ_y is the corresponding Pauli matrix and K is the conjugation operator [36], one can readily show that $\Psi_{n,+}^+ = \Theta\Psi_{n,-}^-$ and $\Psi_{n,-}^+ = \Theta\Psi_{n,+}^-$. As one would expect from the

time reversal invariance $\Psi_{n,+}^-$ is degenerate with $\Psi_{n,-}^+$ and $\Psi_{n,-}^-$ with $\Psi_{n,+}^+$. With these properties in mind the student can guess that expectation values taken with corresponding time reversed wavefunctions must be added together since their energies are degenerate. This fact will have surprising consequences in the next section.

4. Charge and spin currents in the ground state

4.1. Direct calculation

Having found the eigenfunctions, one can now compute equilibrium properties such as persistent charge and spin currents on the ring. The student might be accustomed to expecting a current only when an external force is applied to the system, such as a potential difference. Nevertheless we will show that when symmetry breaking fields exist that can do no work, such as a magnetic field, currents can exist as equilibrium properties borne from the nature of the wavefunctions. In a sense the system can distinguish between clockwise moving current and a counterclockwise moving current, one of them corresponding to a lower energy. These currents are long lived in the absence of perturbations. This is very counterintuitive for the classical line of thought.

There are charge currents in equilibrium only when time reversal symmetry breaking perturbations are present, such as a magnetic field [1,37], for an experimental review see [38] and references therein. The charge currents can be derived directly from the ground state energy as

$$I_{charge} = - \sum_{n,s} \frac{\partial E_{n,s}^\lambda}{\partial \Phi} \quad (36)$$

where Φ is the magnetic flux and the λ quantum number is chosen by the direction of the magnetic field. An alternative form of computing the currents is by using the definition of the current operator. These current densities are fields usually defined from a continuity equation rather than from matrix elements of some operators. In order to generalize the classical definition of the current density $\vec{j}(\vec{r}) = n(\vec{r})e\vec{v}$, we take the expectation value of the velocity operator

$$\vec{J}_{charge} = \Psi^\dagger e \vec{v} \Psi. \quad (37)$$

where e is the electron charge and \vec{v} is the velocity operator. Note that the current is defined in such a way that its dimension is a charge times a velocity. In the presence of the spin-orbit interaction, the velocity operator is not simply \vec{p}/m . There arises what is known as an additional anomalous velocity term. We start from the quantum mechanical definition of the velocity (which is now a 2×2 matrix due to the presence of the spin-dependent terms in the Hamiltonian) $\vec{v} = \frac{i}{\hbar} [\mathbf{H}, \vec{r}]$. Since we are interested in the azimuthal velocity component $\dot{\varphi}$, it is more convenient to calculate

$$\mathbf{v}_\varphi = \frac{ia}{\hbar} [\mathbf{H}, \varphi] = \frac{\hbar}{ima} \partial_\varphi \mathbf{1}_{2 \times 2} - \alpha \sigma_\rho. \quad (38)$$

rather than working in Cartesian coordinates. Note that the Hamiltonian takes a simple form when expressed in terms of the velocity: $\mathbf{H} = \frac{1}{2}m(\mathbf{v}_\varphi^2 - \alpha^2 \mathbf{1}_{2 \times 2})$. Here, we do not consider symmetry breaking magnetic fields, so there can be no persistent charge currents on the ring. If we compute the charge currents for the fourfold degenerate lowest energy levels from which one forms a totally symmetric linear combination, one gets

$$\begin{aligned} J_\varphi = & -\frac{e}{4}(\Psi_{1/2,+}^+)^\dagger \mathbf{v}_\varphi \Psi_{1/2,+}^+ - \frac{e}{4}(\Psi_{1/2,-}^-)^\dagger \mathbf{v}_\varphi \Psi_{1/2,-}^- \\ & - \frac{e}{4}(\Psi_{-1/2,+}^-)^\dagger \mathbf{v}_\varphi \Psi_{-1/2,+}^- - \frac{e}{4}(\Psi_{-1/2,-}^+)^\dagger \mathbf{v}_\varphi \Psi_{-1/2,-}^+ \end{aligned} \quad (39)$$

$$\begin{aligned}
&= -\frac{e\hbar}{4ma} \left[\frac{1}{2} - \frac{1}{2\cos\theta} \right] - \frac{e\hbar}{4ma} \left[-\frac{1}{2} + \frac{1}{2\cos\theta} \right] \\
&- \frac{e\hbar}{4ma} \left[-\frac{1}{2} - \frac{1}{2\cos\theta} \right] - \frac{e\hbar}{4ma} \left[\frac{1}{2} + \frac{1}{2\cos\theta} \right] = 0
\end{aligned} \tag{40}$$

provided the SO strength is within the arrows in Fig. 2, since outside the interval the computation involves a different two-fold degenerate level. The result above applies to any normalized linear combination of the fourfold ground states satisfying the time-reversal symmetry. Charge current cancellation happens level by level and for all α values, rendering them zero as expected. The effect of an external magnetic field (playing the role of a time-reversal symmetry breaking field) and the ensuing charge persistent currents in the presence of a simple scalar potential and the spin-orbit interaction were examined in ref. [18]. The small scalar potential breaks the degeneracy at the edge of the Brillouin zone that limits the charge persistent currents to a maximum value for each band.

On the other hand, there can exist currents that don't break time reversal symmetry i.e. spin currents. In order to make an explicit calculation of the current, here we sidestep the consideration of a scalar potential and use the empty lattice approximation [39]. The spin current density, denoted as $\vec{\mathcal{J}}^a$ in order not to get confused with the charge current density \vec{J} , follows from the same approach as above. Note that the spin current $\vec{\mathcal{J}}^a$ is a tensor (two indices) while \vec{J} is a vector. We define now a local "spin velocity" operator, properly symmetrized

$$\vec{\mathcal{J}}^a = \frac{1}{2} \Psi^\dagger \{ \vec{v}, \mathbf{s}^a \} \Psi, \tag{41}$$

where $\mathbf{s}^a = \hbar \boldsymbol{\sigma}^a / 2$. The reader should note that the time reversal operation applied to such an operator (in the absence of a magnetic field) reverses both the velocity and the spin, so the current is not changed by such an operation, so this intuitive definition materializes our earlier expectation.

The spin current along the ring will again involve the previously derived azimuthal velocity operator (38). The general contribution for any single state involved in the current is given by

$$\begin{aligned}
\mathcal{J}_\varphi^a &= \frac{\hbar^2}{2ma} (\lambda n - 1/2) [|A_{\lambda s}|^2 \sigma_{11}^a + B_{\lambda s}^* A_{\lambda s} e^{-i\varphi} \sigma_{21}^a] \\
&+ \frac{\hbar^2}{2ma} (\lambda n + 1/2) [A_{\lambda s}^* B_{\lambda s} e^{i\varphi} \sigma_{12}^a + |B_{\lambda s}|^2 \sigma_{22}^a] \\
&- \frac{\hbar}{2} \alpha (\cos \varphi \delta_{x,a} + \sin \varphi \delta_{y,a}),
\end{aligned} \tag{42}$$

where $a = x, y, z$ and we have used notation from Eq. (28). Adding contributions from the four degenerate lowest lying levels (again the totally symmetric linear combination) with the spin orientation in the z direction, one arrives at

$$\begin{aligned}
\mathcal{J}_\varphi^z &= \frac{1}{4} (\Psi_{1/2,+}^+)^\dagger \frac{1}{2} \{ \mathbf{v}_\varphi, \mathbf{s}^z \} \Psi_{1/2,+}^+ + \frac{1}{4} (\Psi_{1/2,-}^-)^\dagger \frac{1}{2} \{ \mathbf{v}_\varphi, \mathbf{s}^z \} \Psi_{1/2,-}^- \\
&+ \frac{1}{4} (\Psi_{-1/2,+}^-)^\dagger \frac{1}{2} \{ \mathbf{v}_\varphi, \mathbf{s}^z \} \Psi_{-1/2,+}^- + \frac{1}{4} (\Psi_{-1/2,-}^+)^\dagger \frac{1}{2} \{ \mathbf{v}_\varphi, \mathbf{s}^z \} \Psi_{-1/2,-}^+ \\
&= \frac{\hbar^2}{8ma} \left(\frac{1}{2} \cos \theta - \frac{1}{2} \right) + \frac{\hbar^2}{8ma} \left(\frac{1}{2} \cos \theta - \frac{1}{2} \right) \\
&- \frac{\hbar^2}{8ma} \left(-\frac{1}{2} \cos \theta + \frac{1}{2} \right) - \frac{\hbar^2}{8ma} \left(-\frac{1}{2} \cos \theta + \frac{1}{2} \right),
\end{aligned} \tag{44}$$

$$= \frac{\hbar^2}{4ma} (\cos \theta - 1). \tag{45}$$

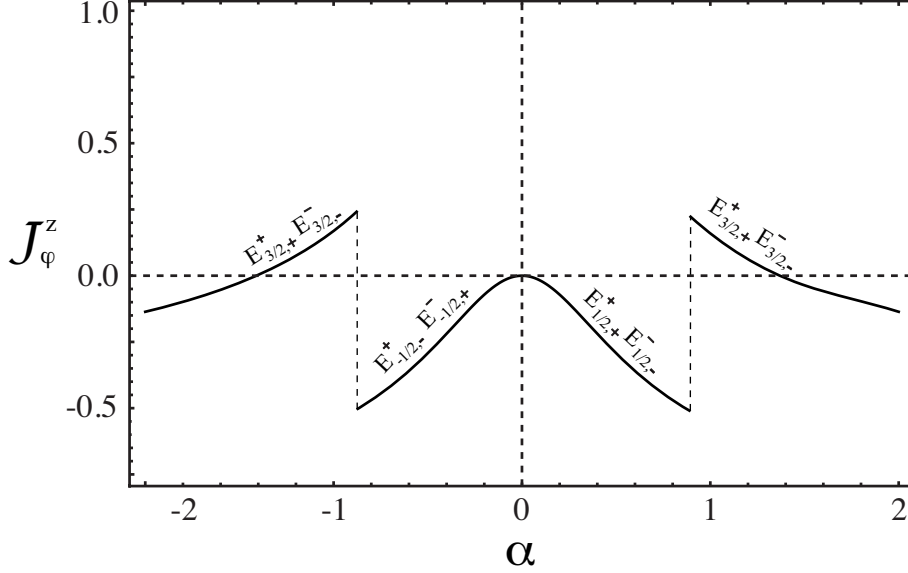


Figure 3. Spin current in units of $\hbar^2/4ma$ from the lowest lying degenerate level indicated in Fig.2 as a function of α . Between the arrows in Fig.2 the lowest level is fourfold degenerate so there are four terms in the spin current. Outside that range, other two-fold degenerate level take over causing a jump in the spin current as indicated. The graph shows how the limit $\alpha = 0$ indicates zero spin current because up down spin symmetry along with inversion symmetry are restored.

Such an expression only accounts for α values in the range between the arrows in Fig.2. Outside this range the energies $E_{3/2,+}^+, E_{3/2,-}^-$ are the lowest energies, so we must compute the expectation of the spin current with such two-fold degenerate eigenfunctions. The full spin current for the lowest lying levels is depicted in Fig.3. One can see from the figure, that at $\alpha = 0$ the spin currents vanish, as required by the recovery of the inversion symmetry ($k \rightarrow -k$) and, independently, the spin inversion symmetry that were intertwined in the presence of the SO interaction. In the presence of a small scalar potential on the ring, the level crossing in Fig.2 gives way to level repulsion. The abrupt transition in dashed lines in Fig.3 will then be gradual and rounded [37]. Persistent spin currents have yet to be measured due to the lack of an appropriate probe.

The spin currents $\mathcal{J}_\varphi^{x,y}$ can also be readily computed along the lines above. In contrast to the spin current \mathcal{J}_φ^z , such currents for each of the degenerate states of a particular level, are not real valued. Nevertheless, the summation of all the degenerate contribution renders a real spin currents which for the ground state is (in the spin-orbit strength range shown in Fig.2)

$$\begin{aligned}\mathcal{J}_\varphi^x &= \frac{1}{2}\hbar\alpha(\cos\theta - 1)\cos\varphi, \\ \mathcal{J}_\varphi^y &= \frac{1}{2}\hbar\alpha(\cos\theta - 1)\sin\varphi.\end{aligned}\tag{46}$$

Such expressions have a φ dependence due to the precession of the spin around the z axis. This means that a particular polarization rotates going through zero and then changing sign, making the whole spin current change sign while the wavevector is constant. Note that the spin persistent currents vanish correctly for $\alpha = 0$ as

expected. Figure 4 shows the angular dependence for both spin current components for a value of α within the range shown in Fig.2.

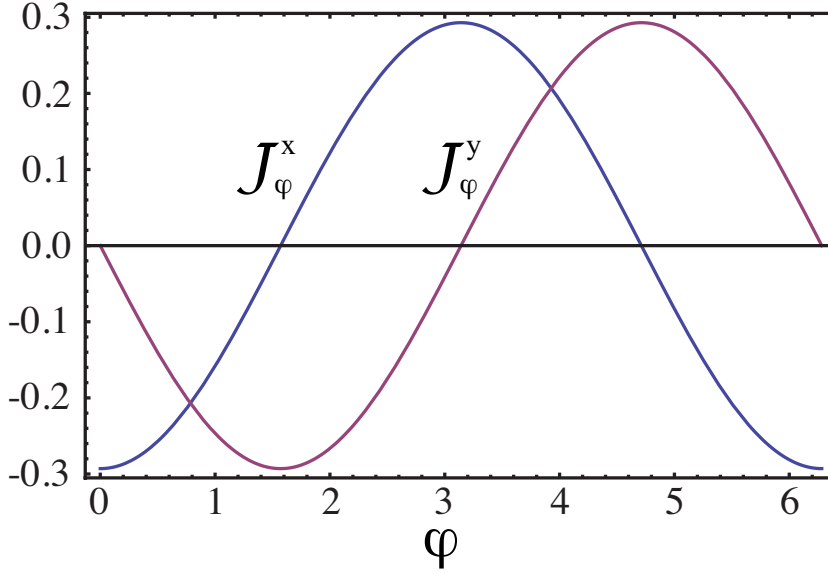


Figure 4. Spin currents for the x, y polarizations in units of $\hbar^2/4ma$ from the lowest lying degenerate levels indicated in Fig.2 as a function of φ and for $\alpha = \frac{\hbar}{2ma}$. Between the arrows in Fig.2, the lowest level is fourfold degenerate so there are four terms in each of the spin currents.

As with charge persistent currents one has to add all occupied levels. Observing figure 2 one sees that the first two levels add constructively while the third level subtracts from the previous two levels. One must then carefully add the corresponding currents taking into account level crossings.

4.2. The Dresselhaus SO interaction

The Dresselhaus Hamiltonian of equation (6) is another form of SO interaction. Using either of the procedures shown in section 2, keeping only p_φ terms, we express $\sigma_x p_x - \sigma_y p_y$ as $-\sigma_x \sin \varphi p_\varphi - \sigma_y \cos \varphi p_\varphi$ that we then symmetrize to get

$$\begin{aligned} \mathbf{V}_{Dresselhaus}^{\text{ring}} &= -\frac{1}{2}\beta(\sigma_x\{\sin \varphi, p_\varphi\} + \sigma_y\{\cos \varphi, p_\varphi\}) \\ &= i\hbar\beta a^{-1}[(\sigma_x \sin \varphi + \sigma_y \cos \varphi)\partial_\varphi + \frac{1}{2}(\sigma_x \cos \varphi - \sigma_y \sin \varphi)], \end{aligned} \quad (47)$$

or we define the combination

$$\mathbf{W} = (\sigma_x \sin \varphi + \sigma_y \cos \varphi)i\partial_\varphi + iA(\sigma_x \cos \varphi - \sigma_y \sin \varphi), \quad (48)$$

and form the Hermitian conjugate

$$\mathbf{W}^\dagger = (\sigma_x \sin \varphi + \sigma_y \cos \varphi)i\partial_\varphi + i(1 - A^*)(\sigma_x \cos \varphi - \sigma_y \sin \varphi), \quad (49)$$

where hermiticity requires $A^* = A = \frac{1}{2}$.

The method of Ref. [33] applied to the Dresselhaus SO interaction would require to express ∂_x and ∂_y in equation (6) in terms of ∂_ρ and ∂_φ , then to substitute ρ by a and ∂_ρ by $-(2a)^{-1}$. It obviously leads to the same expression.

Now, completing the square as we have done in the Rashba case, we get for the Dresselhaus SO interaction the more compact expression

$$\mathbf{H}_{\text{Dresselhaus}}^{\text{ring}} = \frac{\hbar^2}{2ma^2} \left[\left(i\partial_\varphi \mathbf{1}_{2 \times 2} + \frac{ma\beta}{\hbar} (\boldsymbol{\sigma}_x \sin \varphi + \boldsymbol{\sigma}_y \cos \varphi) \right)^2 - \left(\frac{ma\beta}{\hbar} \right)^2 \mathbf{1}_{2 \times 2} \right]. \quad (50)$$

This expression is very similar to the Rashba Hamiltonian and only slight modifications are needed to calculate the spin current along the ring. The energy levels are now given by

$$E_{n,s}^\lambda = \frac{\hbar^2}{2ma^2} \left[\left(n - \frac{\lambda s}{2} \sqrt{1 + 4(ma\beta/\hbar)^2} \right)^2 - (ma\beta/\hbar)^2 \right], \quad (51)$$

i.e. they are obtained from the Rashba eigenenergies via the substitution $\alpha \rightarrow \beta$ and time reversal symmetry is preserved. The corresponding eigenspinors become

$$\Psi_{n,+}^+ = e^{in\varphi} \begin{pmatrix} \sin \frac{\theta}{2} e^{i\varphi/2} \\ -i \cos \frac{\theta}{2} e^{-i\varphi/2} \end{pmatrix}, \quad \Psi_{n,-}^+ = e^{in\varphi} \begin{pmatrix} -\cos \frac{\theta}{2} e^{i\varphi/2} \\ -i \sin \frac{\theta}{2} e^{-i\varphi/2} \end{pmatrix} \quad (52)$$

$$\Psi_{n,+}^- = e^{-in\varphi} \begin{pmatrix} \sin \frac{\theta}{2} e^{i\varphi/2} \\ -i \cos \frac{\theta}{2} e^{-i\varphi/2} \end{pmatrix}, \quad \Psi_{n,-}^- = e^{-in\varphi} \begin{pmatrix} -\cos \frac{\theta}{2} e^{i\varphi/2} \\ -i \sin \frac{\theta}{2} e^{-i\varphi/2} \end{pmatrix} \quad (53)$$

where now $\cos \theta = 1/\sqrt{1 + 4(ma\beta/\hbar)^2}$. When the SO coupling β is not too strong, we recover the structure of a four-fold degenerate ground state and the spin current polarized in the z direction takes the form

$$\mathcal{J}_\varphi^z = \frac{\hbar^2}{4ma} (1 - \cos \theta). \quad (54)$$

Notice that the computed z component of the spin current has the opposite sign to that of the Rashba contribution (see (45)), so that given equal SO strengths for both type of interactions the two effects compete to produce a vanishing net spin current.

4.3. Spin current and the non-Abelian gauge formalism

The computed spin currents in the previous sections were based on intuitive extensions of the very familiar charge current definition, i.e. the physical quantity being transported times the velocity, along with the necessary symmetrization when the definition involves two non-commuting operators. In this section we make the student realize that a formal, well grounded approach can be used to define the currents based on regarding them as conserved quantities following Noether's theorem. We will depart from a general Lagrangian that will be identified with the corresponding ring Hamiltonian, derived above. New tensor fields will arise, analogous to the gauge potential in electromagnetism but of a non-abelian nature, that will by canonical relations, give us the full expressions of the currents in the theory.

The spin current calculated above is the equivalent of the paramagnetic current in the case of $U(1)$ gauge theory (see e.g. [8]). One can ask about a possible equivalent to the diamagnetic contribution also (such a contribution is called color diamagnetism by Tokatly [40, 41]). The Lagrangian approach is very convenient to investigate this question. To the Hamiltonian given in equation (25), we associate a Lagrangian density \mathcal{L} defined according to

$$\begin{aligned} L &\equiv \int ad\varphi \mathcal{L}, \\ &= \langle \Psi | i\hbar \partial_t \mathbf{1}_{2 \times 2} - \mathbf{H} | \Psi \rangle, \\ &= \int ad\varphi [i\hbar \Psi^\dagger \dot{\Psi} - \Psi^\dagger \mathbf{H} \Psi]. \end{aligned} \quad (55)$$

After an integration by parts, \mathcal{L} follows,

$$\begin{aligned} \mathcal{L} = & i\hbar\Psi^\dagger\dot{\Psi} \\ & - \frac{1}{2m} \left[\Psi^\dagger(i\hbar a^{-1}\overleftarrow{\partial}_\varphi \mathbf{1}_{2\times 2} - \frac{1}{2}gW_\varphi^a\boldsymbol{\sigma}_a) \right] \left[(-i\hbar a^{-1}\overrightarrow{\partial}_\varphi \mathbf{1}_{2\times 2} - \frac{1}{2}gW_\varphi^a\boldsymbol{\sigma}_a)\Psi \right] \\ & + \frac{1}{8m}g^2\Psi^\dagger(W_\varphi^b\boldsymbol{\sigma}_b)(W_\varphi^c\boldsymbol{\sigma}_c)\Psi, \end{aligned} \quad (56)$$

where the arrows above the partials indicate in which direction the derivative is taken and the non-Abelian gauge field

$$\frac{1}{2}gW_\varphi^a\boldsymbol{\sigma}_a = m\alpha\boldsymbol{\sigma}_\rho, \quad (57)$$

was introduced with $g = \hbar$ so that W_φ^a has dimensions of $m\alpha/\hbar$. The $SU(2)$ gauge field is then proportional to the SO strength. The contraction on the internal index $a = x, y, z$ is understood. From this Lagrangian, the spin current follows from derivatives with respect to the gauge field components,

$$\mathcal{J}_\varphi^a = \frac{\partial \mathcal{L}}{\partial W_\varphi^a} \quad (58)$$

which has the correct dimension of a spin current. A first contribution $\mathcal{J}_{(1)}$ to the spin current follows from the second term at the r.h.s. of equation (56),

$$\begin{aligned} \mathcal{J}_{(1)}^a = & -\frac{1}{2m}\Psi^\dagger(-\frac{1}{2}\hbar\sigma_a)(-i\hbar a^{-1}\partial_\varphi \mathbf{1}_{2\times 2} - m\alpha\boldsymbol{\sigma}_\rho)\Psi \\ & - \frac{1}{2m}\Psi^\dagger(i\hbar a^{-1}\partial_\varphi \mathbf{1}_{2\times 2} - m\alpha\boldsymbol{\sigma}_\rho)(-\frac{1}{2}\hbar\sigma_a)\Psi, \end{aligned} \quad (59)$$

and can be explicitly calculated for all three spin components,

$$\mathcal{J}_{(1)}^x = -\frac{i\hbar^2}{4ma}[\Psi^\dagger\sigma_x\partial_\varphi\Psi - (\partial_\varphi\Psi^\dagger)\sigma_x\Psi] - \frac{1}{2}\hbar\alpha\Psi^\dagger\cos\varphi\Psi, \quad (60)$$

$$\mathcal{J}_{(1)}^y = -\frac{i\hbar^2}{4ma}[\Psi^\dagger\sigma_y\partial_\varphi\Psi - (\partial_\varphi\Psi^\dagger)\sigma_y\Psi] - \frac{1}{2}\hbar\alpha\Psi^\dagger\sin\varphi\Psi, \quad (61)$$

$$\mathcal{J}_{(1)}^z = -\frac{i\hbar^2}{4ma}[\Psi^\dagger\sigma_z\partial_\varphi\Psi - (\partial_\varphi\Psi^\dagger)\sigma_z\Psi]. \quad (62)$$

The “diacolor” contribution (linear in α) mentioned above appears only for the x and y spin polarizations. There is nevertheless another term in the Lagrangian, the third term of the r.h.s. in equation (56) which plays the role of a *gauge symmetry breaking* term (or a mass term). It brings another contribution $\mathcal{J}_{(2)}$ to the spin current,

$$\mathcal{J}_{(2)}^a = \frac{1}{8m}g^2\Psi^\dagger(\sigma_a W_\varphi^b\sigma_b + W_\varphi^b\sigma_b\sigma_a)\Psi, \quad (63)$$

$$= \frac{1}{4}\hbar\alpha\Psi^\dagger\{\sigma_a, \sigma_\rho\}\Psi, \quad (64)$$

so

$$\mathcal{J}_{(2)}^x = \frac{1}{2}\hbar\alpha\Psi^\dagger\cos\varphi\Psi, \quad (65)$$

$$\mathcal{J}_{(2)}^y = \frac{1}{2}\hbar\alpha\Psi^\dagger\sin\varphi\Psi, \quad (66)$$

$$\mathcal{J}_{(2)}^z = 0, \quad (67)$$

that exactly cancels the diacolor contribution in such a way that for all three spin polarizations only the “paracolor” contributions survive. Eventually the total spin current density, given by

$$\mathcal{J}_\varphi^a = -\frac{i\hbar^2}{4ma}[\Psi^\dagger\sigma_a\partial_\varphi\Psi - (\partial_\varphi\Psi^\dagger)\sigma_a\Psi], \quad (68)$$

is in agreement with the explicit calculations presented above for the z -polarization. In the general case, due to the presence GSB term, it does not coincide with the usual definition of the current in terms of the symmetrized spin-velocity product (41). This property is not very well known and deserves special attention.

5. Path integral on the ring and voltage quantization

Many questions in the context of mesoscopic rings concern phase effects. When a perpendicular magnetic field is enclosed by the ring there occur peculiar conditions to ensure a single-valued wave function [42]. The well known Aharonov-Bohm effect is a striking consequence of the peculiar phase relations on a multiply connected structure such as a ring. In the presence of SO interactions, phase relations are upgraded to spinor interferences since the wavefunctions are now two component objects. Special attention must be paid when a SO interaction is present, since a new physical effect appears i.e. the spin precession. Spin precession is a two component phase evolution in contrast to the one component phase evolution in electromagnetism or $U(1)$ gauge theory.

There is a simple qualitative picture, which consists in the interpretation of the interaction term as similar to an effective Zeeman term, $\sim -\vec{\sigma} \cdot \vec{B}_{\text{eff.}}$. In the case of the Rashba SO interaction, the effective magnetic field is

$$\vec{B}_{\text{Rashba}}^{\text{eff.}} = \alpha p_\varphi \vec{e}_\rho \quad (69)$$

and the axis around which the spin precesses rotates when the electron moves along the ring.

This condition appears when we consider the $SU(2)$ -phase accumulation by the spinor under transport along the loop [43]:

$$\hbar^{-1} \oint \frac{1}{2} g \vec{W}^a \sigma_a d\vec{r} = \frac{m\alpha}{\hbar} \int_0^{2\pi} a d\varphi \sigma_\rho. \quad (70)$$

This expression follows from the gauge formulation of the problem [43, 44], and it is apparent for example in the Rashba case in equation (25) where the spin of the electron appears to be minimally coupled to the electric field through the $SU(2)$ -kinematic momentum

$$(-i\hbar a^{-1} \partial_\varphi) \mathbf{1}_{2 \times 2} - m\alpha \sigma_\rho, \quad (71)$$

where $m\alpha \sigma_\rho \equiv \frac{1}{2} g \vec{W}^a \sigma_a$ is a non-Abelian $SU(2)$ -gauge field introduced in equation (57). When the operator $\exp\left(i\hbar^{-1} \oint \frac{1}{2} g \vec{W}^a \sigma_a d\vec{r}\right)$ constructed from equation (70) is applied on $\Psi = \psi_\uparrow |+\rangle_\rho + \psi_\downarrow |-\rangle_\rho$, the condition

$$\cos \frac{2\pi m\alpha a}{\hbar} \mathbf{1}_{2 \times 2} \Psi + i\sigma_\rho \sin \frac{2\pi m\alpha a}{\hbar} \Psi = \Psi \quad (72)$$

follows to secure single-valued spinors. We thus obtain a quantization condition

$$\frac{2\pi m\alpha a}{\hbar} = 2\pi \times \text{integer}. \quad (73)$$

In the case of the standard SO interaction, $\alpha = |e|\hbar E/2m^2 c^2$. This condition may be rewritten in terms of a typical voltage in the problem defined by $V = 2\pi a E$ (take care here, the electrons on the ring move in a constant potential and the gate voltage applied externally to produce the electric field is a different quantity V_{gate}),

$$\frac{V}{4\pi m c^2 / e} = \text{integer}, \quad (74)$$

where one can introduce a quantum of voltage, $V_0 = 4\pi m c^2 / e$.

The same result also follows from more general arguments based on the path integral formulation on the ring. As the student has learned in graduate quantum mechanics, path integral allow for an alternative and illuminating derivation of quantum mechanics generalizing the action principle of classical mechanics. We will close our description of ring physics in the presence of SO interactions by deriving in detail its path integral and making contact with the quantization conditions derived above.

Let us consider again the Hamiltonian (25) and define the eigenstates $|n\rangle$ of the operator $i\partial_\varphi$, $i\partial_\varphi|n\rangle = -n|n\rangle$, such that periodicity along the ring and normalization are satisfied, $\langle\varphi + 2\pi|n\rangle = \langle\varphi|n\rangle$ and $\int_0^{2\pi} d\varphi |\langle\varphi|n\rangle|^2 = 1$. We have

$$\langle\varphi|n\rangle = (2\pi)^{-1/2} e^{in\varphi}, \quad n \in \mathbb{Z}. \quad (75)$$

In order to construct the amplitude for an electron to go from a particular state on the ring $|\varphi, \sigma; t\rangle$ to another similar state at different time $|\varphi', \sigma'; t'\rangle$, we construct the path integral in discretized time as

$$\mathcal{A}(\varphi, \sigma; t \rightarrow \varphi', \sigma'; t') = \prod_{i=1}^{N-1} \langle\varphi_{i+1}, \sigma_{i+1}| e^{-i\Delta t \mathbf{H}/\hbar} |\varphi_i, \sigma_i\rangle, \quad (76)$$

with $\Delta t = t_{i+1} - t_i$. In the limit $\Delta t \rightarrow 0$, the evolution operator is expanded to linear order in Δt and one has to evaluate matrix elements of the Hamiltonian between spin states. Expanding on the “plane wave” basis, one has

$$i\partial_\varphi \mathbf{1}_{2 \times 2} |\varphi, \sigma\rangle = \sum_{n \in \mathbb{Z}} (-n) (2\pi)^{-1/2} e^{-in\varphi} \mathbf{1}_{2 \times 2} |n, \sigma\rangle, \quad (77)$$

and the matrix element of the Hamiltonian reads as

$$\langle\varphi', \sigma'|\mathbf{H}|\varphi, \sigma\rangle = \frac{\hbar^2}{2ma^2} \sum_{n \in \mathbb{Z}} \frac{1}{2\pi} [n^2 \delta_{\sigma, \sigma'} - 2n \frac{ma\alpha}{\hbar} (\boldsymbol{\sigma}_\rho)_{\sigma, \sigma'}] e^{in(\varphi' - \varphi)}. \quad (78)$$

When we pass to the evolution operator, one has to evaluate

$$\langle\varphi', \sigma'|\mathbf{1}_{2 \times 2} - \frac{i\Delta t}{\hbar} \mathbf{H}|\varphi, \sigma\rangle = \langle\varphi'|\varphi\rangle \delta_{\sigma, \sigma'} - \frac{i\Delta t}{\hbar} \langle\varphi', \sigma'|\mathbf{H}|\varphi, \sigma\rangle, \quad (79)$$

where we make use of the property $\langle\varphi'|\varphi\rangle = \sum_{n \in \mathbb{Z}} \frac{1}{2\pi} e^{in(\varphi' - \varphi)}$ in order to factorize out the term $e^{in(\varphi' - \varphi)}$, and we exponentiate again the Hamiltonian matrix element to get

$$\langle\varphi', \sigma'|\mathbf{1}_{2 \times 2} - \frac{i\Delta t}{\hbar} \mathbf{H}|\varphi, \sigma\rangle = \sum_{n \in \mathbb{Z}} \frac{1}{2\pi} \left[e^{-\frac{i\hbar}{2ma^2} \Delta t [n^2 \mathbf{1}_{2 \times 2} - 2n \frac{ma\alpha}{\hbar} \boldsymbol{\sigma}_\rho]} \right]_{\sigma, \sigma'} e^{in(\varphi' - \varphi)}. \quad (80)$$

Completing the square and introducing the classical variable $v_\varphi = a(\varphi' - \varphi)/\Delta t$, one eventually arrives at

$$\langle\varphi_{i+1}, \sigma_{i+1}| e^{-i\Delta t \mathbf{H}/\hbar} |\varphi_i, \sigma_i\rangle = \text{const} \left[e^{\frac{i}{\hbar} [\frac{1}{2} m (v_\varphi \mathbf{1}_{2 \times 2} - \alpha \boldsymbol{\sigma}_\rho)^2] \Delta t} \right]_{\sigma, \sigma'}, \quad (81)$$

such that extended to the whole path, we get the symbolic expression

$$\mathcal{A}(\varphi, \sigma; t \rightarrow \varphi', \sigma'; t') = \int \mathcal{D}\varphi(t) \mathbf{T} \left[e^{\frac{i}{\hbar} \int \frac{1}{2} m (v_\varphi \mathbf{1}_{2 \times 2} - \alpha \boldsymbol{\sigma}_\rho)^2 dt} \right]_{\sigma, \sigma'}, \quad (82)$$

where \mathbf{T} is a super-operator ordering chronologically all operator products. We note that the path integral representation is only performed at the level of the space degrees of freedom and not for the spin variables for which we still have an evolution operator. Introducing spin coherent states, one could formulate path integrals where

both coordinate and spin would be classical variables [45]. From the path integral, we can read a “classical Lagrangian”

$$\mathbf{L} = \frac{1}{2}m(v_\varphi \mathbf{1}_{2 \times 2} - \alpha \boldsymbol{\sigma}_\rho)^2. \quad (83)$$

When we expand the square, we get three contributions to the spin precession. If we consider the “phase accumulation” for a classical trajectory along a closed path over the ring (one round trip), the first term is simply the dynamical phase, which is path dependent,

$$\frac{1}{\hbar} \int_0^{2\pi} \frac{1}{2} m v_\varphi a d\varphi \delta_{\sigma, \sigma'} = \frac{\pi m v_\varphi a}{\hbar} \delta_{\sigma, \sigma'}. \quad (84)$$

It corresponds to the analogue of the optical phase. The second term, the non-Abelian equivalent to the Aharonov-Bohm phase, is the path independent (for closed paths) contribution discussed in the beginning of this section,

$$\frac{1}{\hbar} \int_0^{2\pi} m v_\varphi \alpha (\boldsymbol{\sigma}_\rho)_{\sigma, \sigma'} dt = \frac{2\pi m \alpha (\boldsymbol{\sigma}_\rho)_{\sigma, \sigma'} a}{\hbar}, \quad (85)$$

or, using α and V defined above,

$$\frac{V}{2mc^2/e} (\boldsymbol{\sigma}_\rho)_{\sigma, \sigma'}. \quad (86)$$

This is the counterpart of the famous Φ/Φ_0 phase in $U(1)$ gauge theory. The third term is an additional path-dependent contribution to the phase which takes its origin in the initial GSB term of equation (56),

$$\frac{1}{\hbar} \int_0^{2\pi} \frac{1}{2} m \alpha^2 (\boldsymbol{\sigma}_\rho^2)_{\sigma, \sigma'} dt = \frac{\pi m \alpha^2 \delta_{\sigma, \sigma'} a}{\hbar v_\varphi}. \quad (87)$$

This last term produces a phase variation which varies *quadratically* with the gate voltage applied.

Let us note the particular role of the GSB term in the Hamiltonian. In equation (82), the transition amplitude is given by a path integral where the action involves a term quadratic in the gauge field. In the simpler case of a particle submitted to the $U(1)$ -electromagnetic gauge field, the corresponding transition amplitude would read as

$$\mathcal{A}(\varphi; t \rightarrow \varphi'; t') = \int \mathcal{D}\varphi(t) e^{\frac{i}{\hbar} \int (\frac{1}{2} m v_\varphi^2 + e v_\varphi A_\varphi) dt}, \quad (88)$$

i.e. it would not include any quadratic term in the $U(1)$ gauge field. The presence of such a quadratic term and its contribution (87) to the phase accumulation is a direct consequence of the GSB term in the SO case.

6. Summary and Conclusions

We have presented an overview of the analytical treatment and role of spin-orbit interactions in the problem of mesoscopic ideal rings. For the benefit of the student, we have placed particular importance to the subtleties of deriving the correct Hamiltonian to the problem when posing it in terms of cylindrical coordinates. Such details had been overlooked in the literature for some time until reference [33] clarified this point. Besides the derivation given in the latter reference we have presented two other appealing approaches that may offer a simpler procedure for more involved problems. Once the correct Hamiltonian was posed we set out to explicitly derive the ground state properties in terms of the charge and spin currents as a function of the spin-orbit

strength, pointing out the symmetries of the problem, and making such symmetries explicit both for the wavefunctions and the spectrum. Careful attention was paid to the correct limiting behavior as the spin-orbit interaction was sent to zero, the level degeneracies and all polarization components of the spin current for the lowest filling of the ring.

Using the Lagrangian formulation of the problems we formally derived the equilibrium currents, making a connection with recently reported *color currents* [40]. Such an approach also permitted assessing the role of the naturally occurring gauge symmetry breaking term in the Rashba Hamiltonian, that is absent from the intuitive formulation of the current in terms of the velocity-spin anticommutator. The results point to a physical distinction of the gauge symmetry breaking contribution in the x, y polarization components of the spin current.

We finally addressed topological considerations on the spin-orbit ring. A voltage quantization condition was derived due to uniqueness of the wavefunction around the ring. A path integral approach to the problem was also formulated in order to formally derive a quadratic in spin-orbit strength contribution to the geometrical phase around the ring.

Every one of the topics discussed has been an opportunity to convey to the student the subtleties of the quantum formulation of the problem of spin transport on mesoscopic rings, detailing the calculations involved and making emphasis on physical insight, and not just bare technicalities.

There are many roads that begin from the material treated here. It is interesting to point out that the exact solutions derived are valid both for the limits: i) Slow rotation of the Rashba magnetic field with respect to a rapid precession of the spin and ii) Rapid rotation of the Rashba field in relation to the precession of the spin. The first is the adiabatic limit where Berry phase effects are dominant, while the latter relates to the sudden perturbation limit. Understanding physically the full range of such behaviors in the presence of a confining potential introduces the physics of lateral subbands that may render many interesting effects for mesoscopic spin-orbit rings as quantum circuit elements.

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